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## ON NECESSARY AND SUFFICIENT CONDITIONS FOR REGULATION OF LINEAR SYSTEMS OVER RINGS\*

E. EMRE†

**Abstract.** Necessary and sufficient conditions are given for regulation of linear systems over rings using observers and causal dynamic state feedback systems with the polynomial fractional representation property. The results are then used to obtain stabilizability conditions for systems over integers, delay-differential systems, systems over polynomial rings, and to obtain conditions to make a 2-D system nonrecursive.

**1. Introduction.** Regulation of linear systems over rings has been considered by several authors. (See Pandolfi [1975], Morse [1976], Sontag [1976], Byrnes [1978], [1979], Kamen and Green [1980], Emre and Khargonekar [1980], Hautus and Sontag [1980] and the references therein.) The first solution to the regulation problem using observers and causal dynamic state feedback for finite free split linear systems over arbitrary commutative rings was given in Emre and Khargonekar [1980], where a theory of observers and coefficient assignment by causal dynamic state feedback was developed. Although the split condition is necessary for regulation via coefficient assignment, it is not necessary for regulation.

The purpose of this paper is to replace the split condition by stabilizability and detectability, which are (as we will show) necessary and sufficient conditions for regulation by observers and causal dynamic state feedback systems satisfying the fractional representation property (a system  $(F, G, H, J)$  is said to satisfy the fractional representation property if and only if its transfer matrix can be expressed as  $PQ^{-1}$ , where  $P, Q$  are polynomial matrices such that  $\det Q = \det(zI - F)$ ).

Recently the concept of detectability has been extended to systems over finitely generated algebras by Hautus and Sontag [1980]. In § 2 of this paper we extend the concepts of stabilizability and detectability to linear systems defined over arbitrary commutative rings and prove that these are necessary and sufficient conditions for regulation by using observers and causal dynamic state feedback systems satisfying the polynomial fractional representation property. (For details of this scheme the reader is referred to Emre and Khargonekar [1980] and also to § 2.) Then in § 3, we use the results of § 2 to obtain stabilizability (also detectability) conditions for systems over polynomial rings, delay-differential systems and systems over integers, we also obtain conditions which make a 2-D system nonrecursive. We also discuss the fact that for the first two cases our detectability result is (essentially) the same as that of Hautus and Sontag [1980].

For general properties and formulations concerning linear systems over commutative rings, the reader is referred to the survey papers Sontag [1976] and Kamen [1978].

**2. Stabilizability and detectability.** In this section we introduce some notation and other preliminaries, and then give necessary and sufficient conditions for regulation using observers and dynamic causal state feedback systems satisfying the polynomial fractional representation property, namely, stabilizability and detectability. We will

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first assume that the state is available and concentrate on causal dynamic state feedback. Then we will explain how the case where the state is not available can be solved using these results (which concern stabilizability) and using observers (detectability).

Throughout the paper,  $k$  denotes an arbitrary but fixed commutative ring with identity.  $k^p$  denotes vectors of size  $p$  with entries in  $k$ . For a given set  $S$ ,  $S[z]$  denotes polynomials in  $z$  with coefficients in  $S$ .  $S^{p \times q}$  denotes  $p \times q$  matrices over  $S$ .  $S((z^{-1}))$  denotes formal power series of the form

$$\sum_{i=l}^{\infty} a_i z^{-i},$$

where  $l$  is an integer and  $a_i$  is in  $S$ . A power series  $a$  in  $S((z^{-1}))$  is *causal* (strictly causal) if and only if  $l \geq 0$  ( $l > 0$ ).  $z^{-1}S[[z^{-1}]]$  denotes the set of strictly causal power series with coefficients in  $S$ . For a  $p \times p$  nonsingular matrix  $Q$  over  $k[z]$ ,  $k_Q$  is defined to be the  $k$ -linear module of polynomial vectors  $x$  in  $k^p[z]$  such that  $Q^{-1}x$  is strictly causal (as a power series).

The  $k$ -linear maps  $\Pi$  and  $\Pi_Q$  are defined as follows:

$$\begin{aligned} \Pi: k^p((z^{-1})) &\rightarrow z^{-1}k^p[[z^{-1}]], & x &\mapsto \text{the strictly causal part of } x, \\ \Pi_Q: k^p[z] &\rightarrow k_Q, & x &\mapsto Q\Pi(Q^{-1}x). \end{aligned}$$

For a  $p \times r$  polynomial matrix  $\Phi$  with the  $i$ th column  $\phi_i$ , we define  $\Pi_Q(\Phi)$  to be the  $p \times r$  matrix whose  $i$ th column is  $\Pi_Q(\phi_i)$ . For a  $k$ -linear map  $M: X_1 \rightarrow X_2$ , where  $X_1, X_2$  are  $k$ -linear modules,  $\text{Im } M$  denotes the image of  $X_1$  under  $M$  as a  $k$ -linear module, and  $\text{ker } M$  denotes the kernel of  $M$ . If  $P$  is a  $p \times m$  polynomial matrix whose  $i$ th column is expressed as

$$p_i = \sum_{j=0}^{v_i} a_{ij} z^j,$$

where  $a_{i0} \neq 0$ , we say that  $P$  is *column proper* if and only if  $a_{1v_1}, \dots, a_{mv_m}$  is a set of generators for the  $k$ -module  $k^p$ .  $P$  is *row proper* if and only if its transpose is column proper.

Throughout the paper we assume that there exists a multiplicatively closed set of monic polynomials  $P_s$  in  $k[z]$ , called the set of *stable polynomials*. A rational function  $p/q$ , where  $p, q$  are in  $k[z]$ , is *stable* if and only if  $q$  is in  $P_s$ . A rational matrix is *stable* if and only if all of its entries are stable.

A finite free linear system over a commutative ring  $k$  is the triple  $(F, G, H)$ , where  $F$  is in  $k^{n \times n}$ ,  $G$  is in  $k^{n \times m}$ , and  $H$  is in  $k^{p \times n}$ . Throughout the paper we will be concerned with such systems only. An equivalent representation is in terms of  $k$ -linear maps with a finite free state module. For a detailed introduction to linear systems over rings we refer to the survey papers Sontag [1976] and Kamen [1978].

For a given pair  $(F, G)$ , we define

$$W_i := \text{Im } G + \dots + \text{Im } F^i G, \quad i = 0, 1, \dots$$

For a matrix  $A$ ,  $\det A$  denotes the determinant of the matrix  $A$ .

A system  $(F_1, G_1, H_1, J_1)$  is said to have the *polynomial fractional representation property* if and only if its transfer matrix  $H_1(zI - F_1)^{-1}G_1 + J_1$  can be expressed as

$$P_c Q_c^{-1},$$

where  $P_c, Q_c$  are polynomial matrices (over  $k[z]$ ) such that  $\det Q_c = \det(zI - F_1)$ .

For a matrix  $A$ ,  $\text{Sp}_k A$  denotes the  $k$ -linear module generated by the columns of  $A$ . For a polynomial matrix  $P$ ,  $\delta_{ei}(P)$  denotes the degree of the  $i$ th column of  $P$ .

Now we state the main results of this section.

DEFINITION 2.1. Let  $F$  be in  $k^{n \times n}$ , and let  $G$  be in  $k^{n \times m}$ . Then  $(F, G)$  is *stabilizable* if and only if there exist stable rational matrices  $V_1, V_2$  such that

$$(2.2) \quad [zI - F, G] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = I_n.$$

Remark 2.3. We will call  $(H, F)$  *detectable* if and only if  $(F', H')$  is stabilizable. (For a matrix  $A$ ,  $A'$  denotes the transpose of  $A$ .)

THEOREM 2.4. There exist polynomial matrices  $P_c, Q_c$  such that

- (i)  $Q_c$  is column proper;
- (ii)  $P_c Q_c^{-1}$  is well defined as a power series, and is causal, and has a realization  $(F_1, G_1, H_1, J_1)$  such that  $\det Q_c = \det (zI - F_1)$ ; and
- (iii) the determinant of

$$(2.5) \quad \Phi := (zI - F)Q_c + GP_c$$

is a stable polynomial if and only if  $(F, G)$  is stabilizable.

Proof. Necessity. Postmultiply both sides of (2.5) by  $\Phi^{-1}$ .

Sufficiency. If  $(F, G)$  is stabilizable, then there exist stable polynomial matrices  $V_1, V_2$  satisfying (2.2). Express  $V_1, V_2$  as

$$V_1 = N_1(d \cdot I)^{-1}, \quad V_2 = N_2(d \cdot I)^{-1},$$

where  $N_1, N_2$  are polynomial matrices and  $d$  is a stable monic common multiple of the denominators of the entries of  $V_1$  and  $V_2$ . (Such a  $d$  exists as  $V_1$  and  $V_2$  are both stable.) Then we have

$$(zI - F)N_1 + GN_2 = d \cdot I.$$

Let  $v$  be the smallest integer such that  $W_{v-1} = W_{n-1}$ . Let  $r$  be the degree of  $d$ . Let  $\gamma_i$  be the smallest integer such that

$$(2.6) \quad \gamma_i := lr \geq v$$

for some integer  $l \geq 1$ . Define

$$(2.7) \quad d_1 := d^l, \quad \bar{N}_1 := d^{l-1} \cdot N_1, \quad \bar{N}_2 := d^{l-1} \cdot N_2.$$

Then we have

$$(2.8) \quad (zI - F)\bar{N}_1 + G\bar{N}_2 = d_1 \cdot I.$$

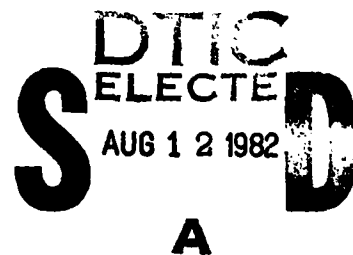
Note here that, as  $P_c$  is multiplicatively closed,  $d_1$  is stable.

Equation (2.8) implies that

$$(2.9) \quad \Pi_{(zI-F)}(d_1 \cdot I) \subset W_{n-1}.$$

Then, by Emre [1980, Thm. 3.1], there exist polynomial matrices  $P_c, Q_c$  such that:

- (i)  $Q_c$  is column proper with the  $i$ th column degree  $\gamma_i - 1$ , and with the highest degree column coefficient matrix  $I$  (which ensures that  $Q_c^{-1}$  is well defined).
- (ii)  $P_c Q_c^{-1}$  is well defined and causal, and has a realization  $(F_1, G_1, H_1, J_1)$  such that  $\det Q_c = \det (zI - F_1)$ . (The fact that this system has the polynomial fractional representation property is seen from the results of Emre and Khargonekar [1980].)



(iii)

$$(zI - F)Q_c + GP_c = d_1 \cdot I.$$

As  $d_1^* = \det(d_1 \cdot I)$  is stable, the proof is complete.  $\square$

The next theorem shows that stabilizability is a necessary and sufficient condition for regulation of the system  $(F, G, I)$  by causal dynamic feedback with the polynomial fractional representation property.

**THEOREM 2.10.** *Let  $F, G$  be given. Then there exists a finite free dynamic feedback system  $(F_1, G_1, H_1, J_1)$  over  $k$  such that*

- (i)  $H_1(zI - F_1)^{-1}G_1 + J_1$  can be expressed as  $P_c Q_c^{-1}$  for some polynomial matrices  $P_c, Q_c$  with the property that  $\det Q_c = \det(zI - F_1)$ , and
- (ii) the characteristic polynomial of the closed loop system obtained by taking the state as the external direct sum of the states of the open loop system and the feedback system is stable if and only if  $(F, G)$  is stabilizable.

*Proof.* Under the hypotheses of the theorem the characteristic polynomial of the closed loop system can be easily shown to be equal to

$$\det((zI - F)Q_c + GP_c).$$

The rest follows from Theorem 2.4.  $\square$

The next theorem provides a criterion to determine the stabilizability of  $(F, G)$  in terms of  $[zI - F, G]$ .

We consider  $k[z, z_1]$ , and its maximal ideals which we denote as  $\{m_\lambda\}_{\lambda \in \Lambda}$  for some index set  $\Lambda$ .

For a matrix  $A = (a_{ij})$  over  $k[z, z_1]$ ,  $A_\lambda$  denotes the matrix which is obtained from  $A$  by replacing  $a_{ij}$  with the residue class of  $a_{ij}$  modulo  $m_\lambda$ . For a detailed description of these concepts, the reader is referred to an algebra book (e.g., Bourbaki [1972, Chapt. 2]). Let  $\{m_\lambda\}_{\lambda \in \Lambda}$  be the set of maximal ideals of  $k[z, z_1]$  such that

$$(2.11) \quad \text{rank}[zI - F, G]_\lambda < n.$$

After these preliminaries, we have:

**THEOREM 2.12.**  *$(F, G)$  is stabilizable if and only if there exists a stable polynomial  $q$  such that*

$$q_\lambda = 0_\lambda$$

for each  $m_\lambda$ .

*Proof. Necessity.* If  $(F, G)$  is stabilizable, by Theorem 2.4 there exist polynomial matrices  $P_c, Q_c, \Phi$ , with  $\det \Phi$  stable, such that (2.5) is satisfied. Then evaluating both sides of (2.5) at each  $m_\lambda$ , we have from (2.11) that  $\det \Phi$  evaluated at each  $m_\lambda$  must be zero.

*Sufficiency.* It follows from Bourbaki [1972, Chapt. 2] that a matrix  $M$  over  $k[z, z_1]$  is right invertible over  $k[z, z_1]$  if and only if  $M_\lambda$  is right invertible over  $k[z, z_1]/m_\lambda$  for each maximal ideal  $m_\lambda$ . Now define

$$M := [zI - F, G, (z_1q - 1) \cdot I].$$

From (2.11), the only maximal ideals such that rank of  $M_\lambda$  can possibly be less than  $n$  are  $\{m_\lambda\}$ . But, for each  $m_\lambda$ , we have  $q_\lambda = 0$ . Hence for each  $m_\lambda$ , we have

$$\text{rank } M_\lambda = n.$$

Thus  $M$  is right invertible over  $k[z, z_1]$ . That is, there exist polynomial matrices  $M_1(z, z_1), M_2(z, z_1), M_3(z, z_1)$  such that

$$(zI - F)M_1 + GM_2 + (z_1q - 1)M_3 = I.$$

But then, letting  $z_1 = 1/q$ , we obtain

$$[zI - F, G] \begin{bmatrix} M_1(z, 1/q) \\ M_2(z, 1/q) \end{bmatrix} = I.$$

As  $q$  is stable, by definition,  $(F, G)$  must be stabilizable.  $\square$

**Remark 2.13.** If  $k = K[s_1, \dots, s_N]$ , where  $K$  is a field, evaluations at the maximal ideals of  $k[z, z_1]$  become evaluations of the polynomials at the points  $(s_1^*, \dots, s_N^*, z^*, z_1^*)$  of  $\bar{K}^{N+2}$ , where  $\bar{K}$  is the algebraic closure of  $K$ . For a detailed discussion of this the reader is referred to Hautus and Sontag [1980], and for further details to Bourbaki [1972, Chapt. 2]. In this case our definition of detectability becomes the same (essentially) as the one developed in Hautus and Sontag [1980]. We should note here that Theorem 2.12 remains valid when  $zI - F$  and  $G$  are replaced by arbitrary polynomial matrices of compatible dimensions.

Based on Theorems 2.4, 2.12, we obtain the following corollary:

**COROLLARY 2.14.** If  $k = K[s_1, \dots, s_N]$ , then  $(F, G)$  is stabilizable if and only if there exists a stable polynomial  $q$  in  $k[z]$  which vanishes at the points of  $\bar{K}^{N+1}$ , where  $[zI - F, G]$  loses rank.

*Proof.* If we note that evaluating a polynomial in  $k[z]$  at the points of  $\bar{K}^{N+2}$  is the same as evaluating it at the points of  $\bar{K}^{N+1}$ , the result follows from Theorem 2.12 and Corollary 2.14.  $\square$

**REMARK 2.15.** If a system is given in the form  $(F, G, H)$  (i.e., the state is not available), then one can use observers and dynamic feedback compensators together, as shown in Emre and Khargonekar [1980], to achieve regulation. It is seen from the formulations given in that paper, and in Hautus and Sontag [1980], that an observer exists if and only if  $(H, F)$  is detectable. Furthermore, in such a scheme, the characteristic polynomial of the closed loop system is the product of the characteristic polynomial of the observer and  $\det((zI - F)Q_c + GP_c)$ , where  $Q_c, P_c$  are as defined in this paper. Hence, regulation can be achieved by using observers and causal dynamic feedback systems having the polynomial fractional representation property if and only if  $(F, G)$  is stabilizable and  $(H, F)$  is detectable, and this result is valid for systems over arbitrary commutative rings with our definitions here.

As for the polynomial fractional representation requirement of the feedback systems, this is not a big restriction as far as known results are concerned because, for example, nondynamic (constant) state feedback satisfies this property trivially. One advantage of this property is that it allows the consideration of internal stability in terms of the polynomial equations arising in stabilizability and detectability and immediately guarantees the realizability of the feedback system. For a natural realization that can be used to implement  $P_c Q_c^{-1}$ , the reader is referred to Kalman, Falb and Arbib [1969], Fuhrmann [1976] and Emre [1980b].

**3. Stabilizability of some specific classes of systems.** In this section, using the results of § 2, we obtain stabilizability (detectability) criteria for certain specific classes of linear systems over rings.

1) *Systems over integers.* These systems are discrete time systems  $(F, G, H)$  over integers. The problem is to construct dynamic compensators with integer coefficients such that the closed-loop system is regulated. In this case the set of stable polynomials is of the form  $z^r$  for some  $r \geq 0$ . From the definition,  $(F, G)$  is stabilizable if and only if  $[zI - F, G]$  has a right inverse whose entries have denominators of the form  $z^r$  for some integer  $r \geq 0$ , or if and only if there exist polynomial matrices  $N_1, N_2$  with integer

coefficients and some integer  $r \geq 0$  such that

$$(3.1) \quad (zI - F)N_1 + GN_2 = z^r \cdot I.$$

We see that (3.1) is possible if and only if

$$(3.2) \quad F' \subset \text{Im } G + \dots + \text{Im } F^{n-1}G,$$

for some  $r \geq 0$ .

2) *2-D systems*. In this case,  $k$  = the ring of proper rational functions over a field. Here the problem is to find a causal dynamic feedback system such that the characteristic polynomial of the closed loop system becomes  $z^r$  for some  $r \geq 0$ . This problem will have a solution if and only if  $(F, G)$  and  $(F', H')$  satisfy the condition (3.2).

3) *Systems over a polynomial ring*  $K[s_1, \dots, s_N]$ . In this case we obtain the following theorem.

**THEOREM 3.3.**  $(F, G)$  is stabilizable if and only if at every point  $(s_1^*, \dots, s_N^*, z^*)$  of  $\bar{K}^{N+1}$  where  $[zI - F, G]$  loses rank, the real part of  $z^*$  is negative.

4) *Delay-differential systems*. This is the same as systems over polynomial rings except that the set of stable polynomials is different. We have Corollary 2.14.

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